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Elementary bound states for the power-law potentials

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Abstract. We show how the exact polynomial $\times$exp(polynomial) bound-state solutions to the radial Schrödinger equation may be constructed for any potential of the form

$$V(r)=\sum_{\delta\in D} G_\delta r^\delta$$

where D is a finite set of arbitrary rational numbers and some of the couplings G_δ are not independent.

1. Introduction

The radial Schrödinger equation

$$-\frac{d^2}{dr^2}\psi(r)+\frac{\lambda^2-\frac{1}{4}}{r^2}\psi(r)+V(r)\psi(r)=E\psi(r) \tag{1}$$

$$\lambda=l+1/2, \qquad l=0,1,\ldots.$$

with the simple harmonic-oscillator potential

$$V(r)=\mu+\nu^2r^2 \tag{2}$$

and its anharmonic generalisations

$$\mu=\mu^{(q)}(r)=\sum_{j=1}^{q}\alpha_j r^{2j-2}, \qquad \nu=\nu^{(q)}(r)=\sum_{j=1}^{q}\beta_j r^{2j-2} \tag{3}$$

$$V(r)=\mu^{(q)}(r)+\nu^{(q)2}(r)r^2=\sum_{j=0}^{2q-1} g_j r^{2j}, \qquad g_{2q-1}=\beta_q^2>0,$$

may be used not only in the quantum mechanics ($V(r)$ represents an arbitrary analytic potential in the limit $q\to\infty$) but also, e.g., in the Reggeon field theory on the lattice (Fulco and Masperi 1979, $q=2$). In these applications, the available numerical or perturbative solution methods are not always sufficient to specify the limitations of validity of the approximations (especially the convergence of the perturbation expansions) and to describe the analytic continuations and other interesting qualitative features of the solution (cf, e.g., Simon 1969).

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When $q \geqslant 2$, not all of the merits of (1) as a simple model are lost. The first non-trivial ($q=2$, sextic) anharmonicity (3) was shown by Singh *et al* (1978) to lead to the complete analytic solution of (1) where $\psi = \exp(-\text{polynomial}) \times \text{power series}$, and $E =$ the root of a continued fraction. The $q>2$ cases of (1)+(3) were solved later in a similar way (Znojil 1981).

Even when $q=2$, the structure of the exact solution of (1) remains rather complicated. Fortunately, provided that we fix properly one of the couplings g and consider the particular, exceptionally simple, solutions only, it is possible to replace both the continued fraction form of E and the infinite power series in ψ by polynomials of degree N and $2N-2$, respectively. Singh *et al* (1978, $q=2$) constructed the corresponding N terminating (exact and elementary) eigenstates of the sextic anharmonic oscillator in an explicit way.

In a similar spirit, Flessas and Das (1980) and Magyari (1981) conjectured an application of the termination requirement to any $q \geqslant 2$ and constructed examples of an exact eigenstate for $N=1$ and 2 and for $q=3$. The former article also inspired our present considerations: In brief, we shall be interested in the particular terminating solutions to the Schrödinger equation (1) with the arbitrary rational power-law interaction

$$V(r) = \sum_{z \in D} G_z r^z \tag{4}$$

($D =$ finite set of the rational numbers) since it generalises the 'solvable' interactions (3) and contains also the singular components with $z<-2$. In more detail, we are motivated by the following physical, formal and methodical respective reasons:

(i) *A priori*, any phenomenological interaction $V(r)$ may be singular in the origin. In the nucleon–nucleon or inter-molecular interactions, the singular components of $V(r)$ (short-range 'core' with $z<-2$ in (4)) represent at least as important a 'realistic' modification of (2) as the more or less arbitrary asymptotic confinement introduced by (3). In this context, the exceptional exact and elementary solution, if available, could serve as a simple quantitative model, or at least as a useful device for testing the various practically oriented approximations and (possibly not fully rigorous) regularisation schemes.

(ii) The transition (2)→(3) does not incorporate the third 'natural' replacement of the parameter λ by a polynomial

$$\lambda^{(p)}(1/r) = \sum_{j=0}^{p} \gamma_j r^{-2j}.$$

Such a change would lead to the more consequent singular-anharmonic generalisation

$$V(r) = \sum_{i=0}^{2q-1} g_i r^{2i} + \sum_{j=1}^{2p+1} h_j r^{-2j}$$

$$g_{2q-1} > 0, \qquad h_{2p+1} > 0, \qquad q \geqslant 1, \qquad p \geqslant 0 \tag{5}$$

of (2). Here, the centrifugal part of (1) may be included into $h_1 = h_1' + l(l+1)$.

(iii) An extension of the idea of Singh *et al* (1978) to the potentials (5) and (4) seems to be straightforward. Moreover, the termination method itself might provide a useful guide to its possible non-termination generalisations in the future.

2. Results

2.1. Canonical form of the potential

The form (5) of the potential is universal—it is equivalent to the general choice of forces (4) with an arbitrary finite set of the rational exponents D. Indeed, the corresponding new Schrödinger equation with the potential (4) may be written in the form

$$-\frac{d^2}{dx^2}\chi(x)+W(x)\chi(x)=\eta\chi(x)$$

$$W(x)=\frac{1-t^2+4h_1}{4t^2x^2}+\sum_{j=1}^{2p}\frac{h_{j+1}}{t^2x^{2+2j/t}}+\sum_{i=1}^{t-1}\frac{g_{i-1}}{t^2x^{2(t-i)/t}}+\sum_{i=t+1}^{2q}\frac{g_{i-1}}{t^2}x^{2(i-t)/t},\qquad \eta=-\frac{g_{t-1}}{t^2} \tag{6}$$

with the properly chosen non-negative integer parameters p, $q(\neq 0)$ and $t(\neq 0)$. Its equivalence to the polynomially anharmonic and singular oscillator (1)+(5) follows simply from the change of variables

$$x=r^t,\qquad \chi(x)=r^{(1-t)/2}\psi(r) \tag{7}$$

with $1\leqslant t\leqslant 2q$ and $g_0=-E$. Obviously, for different t's, the energy η coincides with the different g's. Keeping this in mind, we may often merely consider $t=1$ without loss of generality. It is also worth mentioning that the choice of a negative t, $1\leqslant -t\leqslant 2p$ is possible. This reflects a purely formal $r\leftrightarrow 1/r$, $p\leftrightarrow q$ and $h\leftrightarrow g$ symmetry of the Schrödinger equation (6).

2.2. Factorisation of the wavefunction

When we consider the asymptotic ($r\to\infty$) and threshold ($r\to 0$) regions, we may try to represent the solution of (6) or rather of its canonical form (1)+(5) via a factorisation

$$\psi(r)=\exp\Big(-\sum_{i=1}^{q}\beta_i r^{2i}/2i-\sum_{j=1}^{p}\gamma_j r^{-2j}/2j\Big)\varphi(r)$$

$$\varphi(r)=\sum_{n=N_1}^{N_2}a_n r^{2n+2s} \tag{8}$$

where $N_1=-\infty$ and $N_2=+\infty$ in general. Of course, we have to choose

$$\beta_q=g_{2q-1}^{1/2},\qquad \gamma_p=h_{2p+1}^{1/2}$$

$$\beta_m=\frac{1}{2\beta_q}\Big(g_{q+m-1}-\sum_{k=m+1}^{q-1}\beta_{m+q-k}\beta_k\Big),\qquad m=q-1,q-2,\ldots 1 \tag{9}$$

$$\gamma_n=\frac{1}{2\gamma_p}\Big(h_{p+n+1}-\sum_{j=n+1}^{p-1}\gamma_{n+p-j}\gamma_j\Big),\qquad n=p-1,p-2,\ldots 1$$

which is compatible with the special $g\leftrightarrow\alpha,\beta$ transformation in (3), and with $\varphi(r)=\exp(O(r^2)+O(r^{-2}))$ for $r\to 0$ and $r\to\infty$.

Provided that $\varphi(r)$ degenerates to a polynomial ($N_1>-\infty$ and $N_2<+\infty$) for some particular potential, energy and $p\geqslant 1$, the physical boundary conditions are satisfied in a trivial way, as a consequence of the termination of $\varphi(r)$. For $p=0$, the regular behaviour of $\psi(r)$ in the origin has to be guaranteed by an independent though standard requirement $2(N_1+s)>\frac{1}{2}$.

2.3. Selfconsistent construction of an exact and elementary bound state (main result). When we insert (8) and (5) into (1), we arrive at the recurrences

$$\sum_{i=0}^{q-1}[-g_i-4\beta_{i+1}(n+s-i)-G_i]a_{n-i}$$
$$+\left[-h_1+(2n+2s+2)(2n+2s+1)-2\sum_{k=1}^{\min(p,q)}\beta_k\gamma_k\right]a_{n+1}$$
$$+\sum_{j=2}^{p+1}[-h_j+4\gamma_{j-1}(n+s+j)-H_j]a_{n+j}=0, \tag{10}$$
$$G_i=(2i+1)\beta_{i+1}-\sum_{k=1}^{i}\beta_k\beta_{i-k+1}+2\sum_{m=1}^{\min(p,q-i-1)}\gamma_m\beta_{m+i+1}$$
$$H_j=(2j-1)\gamma_{j-1}-\sum_{k=1}^{j-2}\gamma_k\gamma_{j-k-1}+2\sum_{m=1}^{\min(q,p+1-j)}\beta_m\gamma_{m+j-1}$$

for the Taylor coefficients. Here, n runs fron N_1-p-1 up to N_2+q-1 and

$$a_{N_1-j}=0, \qquad j=1,2,\ldots Q, \qquad Q=p+q \tag{11a}$$

and

$$a_{N_2+i}=0, \qquad i=1,2,\ldots Q. \tag{11b}$$

The first and last row of (10) are mere definitions

$$\begin{aligned} h_{p+1}&=4\gamma_p(N_1+s)-H_{p+1}, \qquad p\neq 0\\ &=2(N_1+s)(2N_1+2s-1), \qquad p=0 \end{aligned} \tag{12}$$

$$g_{q-1}=-4\beta_q(N_2+s)-G_{q-1} \tag{13}$$

of the constants h_{p+1} and g_{q-1}, respectively, while the remaining $K=N+Q-2=N_2-N_1+1+Q-2$ rows have the matrix form

$$Z\begin{pmatrix} a_{N_1}\\ a_{N_1+1}\\ \vdots\\ a_{N_2}\end{pmatrix}=0 \tag{14}$$

of an overcomplete system of the algebraic equations. They define not only the N Taylor coefficients a_n but also the selfconsistent energy and $Q-2$ couplings in (5). The $K\times N$-dimensional matrix Z has $Q+1$ diagonals, denoted

$$\begin{aligned} &Z_{mm+1}=C_m^{(-1)}=B_m, && m=1,2,\ldots N-1\\ &Z_{m+im}=C_{m+i}^{(i)}, && m=1,2,\ldots N, \qquad i=0,1,\ldots,Q-2\\ &Z_{m+Q-1m}=A_{m+Q-1}, && m=1,2,\ldots,N-1 \end{aligned} \tag{15}$$

the explicit form of which follows easily from the comparison with (10).

The simplest non-trivial example of our approach with $t=p=q=1$, degree $N=1$ ($N_1=N_2$) and even 'parity' $2s=0$ leads to the three equations (12)–(14) for the energy and four couplings. Their simple solution reads

$$h_1 = h_1' + l(l+1) = 2N_1(2N_1-1) - 2(g_1 h_3)^{1/2}$$
$$h_2 = (4N_1-3)h_3^{1/2}$$
$$E = -g_0 = (4N_1+1)g_1^{1/2}$$

and defines the potential (5), bound state (8) and its energy E in terms of the free coupling constants $g_1 > 0$ and $h_3 > 0$.

2.4. *Ground states* ($N=1$)

When we choose $N=1$, the function $\varphi(r) = a_1 r^{2s+2}$ in (8) has no nodal zeros and $\psi(r)$ or $\chi(x)$ is a ground state. From (10) we get the requirements

$$B_0 = 0, \qquad C_{k+1}^{(k)} = 0, \qquad k = 0, 1, \ldots, Q-2, \qquad A_Q = 0 \quad (16)$$

which represent an explicit solution of the selfconsistency restrictions (12), (13) and (14). For $p=0$, this solution degenerates to the decoupled one-to-one relations

$$\alpha_{i+1} = -\beta_{i+1}(4s+2i+5), \qquad i = 0, 1, \ldots, q \quad (17)$$

between the couplings defined in (3). It preserves the form of (13), reproduces and simplifies the explicit $N=1$, $q=2$, 3, $t=1$ results of Flessas and Das (1980) and Magyari (1981) and generalises them to any q and t in the regular ($p=0$) polynomially anharmonic oscillator potential $W(x)$ in (6).

When $p \neq 0$, we may introduce the new couplings ρ and δ instead of g (or $\beta(g)$) and h (or $\gamma(h)$), by the prescription

$$\begin{aligned} \rho_i &= h_{p+1-i} + H_{p+1-i}(\beta, \gamma), \qquad & \delta_i &= -4\gamma_{p-i}, \qquad & i &= 0, 1, \ldots, p \\ \rho_{p+j} &= g_{j-1} + G_{j-1}(\beta, \gamma), \qquad & \delta_{p+j} &= 4\beta_j, \qquad & j &= 1, 2, \ldots, q. \end{aligned} \quad (18)$$

Then, an insertion of the formulae

$$\begin{aligned} C_n^{(i)} &= \rho_{i+1} + \delta_{i+1}(n+s-i), \qquad i \neq p-1 \\ C_n^{(p-1)} &= \rho_p - 2(n+s)(2n+2s-1) \end{aligned} \quad (19)$$

into (16) enables us to preserve the full analogy with $p=0$ and formal symmetry between the ρ's and δ's in the $N=1$ states.

2.5. *Multiplets of states*

When we interpret (11*a*) as a initialisation of the recurrences (10), we may easily verify that the Taylor coefficients a_n in (8) or (14) may be given the compact form

$$a_{N_1+n} = (-1)^n a_{N_1} \det Z(n)/(B_1 B_2 \cdots B_n)$$

$$Z(n) = \begin{pmatrix} Z_{11} & Z_{12} & \cdots & Z_{1n} \\ \vdots & & & \\ Z_{n1} & \cdots & & Z_{nn} \end{pmatrix} = \begin{pmatrix} C_1^{(0)} & B_1 & 0 & \cdots & 0 \\ \vdots & & & & \\ \cdots & & & & C_n^{(0)} \end{pmatrix} \quad (20a)$$

where a_{N_1} is an arbitrary normalisation $(=1)$ and $n=1, 2, \ldots N-1$. In this formulation, the remaining $n=N, N+1, \ldots$ items (rows in (14)) are equivalent to the termination requirements (11*b*) and may be replaced by the explicit determinantal selfconsistency conditions

$$\det Z(N-1+n)=0, \qquad n=1, 2, \ldots, Q-1. \tag{21a}$$

Also equation (13) may be identified with (21*a*) at $n=Q$.

Alternatively, we may replace (20*a*) by the analogous formula

$$a_{N_2-n}=(-1)^n a_{N_2} \det Z^*(n)/(A_K A_{K-1} A_{K-n+1}) \tag{20b}$$

$$Z^*(n)=\begin{pmatrix} Z_{KN} & Z_{KN-1} & \cdots & Z_{KN-n+1} \\ \vdots & & & \\ Z_{K-n+1N} & \cdots & Z_{K-n+1N-n+1} & \end{pmatrix}=\begin{pmatrix} C_K^{(Q-2)} A_K & 0 & \cdots & 0 \\ \vdots & & & \\ \cdots & & & C_{K-n+1}^{(Q-2)} \end{pmatrix}$$

with the same range of n and, say, $a_{N_2}=1$. The role of the selfconsistency (termination) requirement is now played by (11*a*), i.e.,

$$\det Z^*(N-1+n)=0, \qquad n=1, 2, \ldots, Q-1 \tag{21b}$$

which reproduces (12) at $n=Q$. For simplicity, we may shift s in such a way that $N_1=1$ and $N_2=N$ in (20).

Among the $(2Q-1)$ constants $g(=g_0, g_1, \ldots, g_{q-2})$, $h(=h_1, h_2, \ldots, h_p)$, $\beta(=\beta_1, \beta_2, \ldots, \beta_q)$ and $\gamma(=\gamma_1, \gamma_2, \ldots, \gamma_p)$, the $Q-1$ selfconsistency relations (21*a*) or (21*b*) are to be satisfied. The choice of the variables to be fixed is arbitrary. For both the physical and formal reasons, it is preferable to pick up the ρ's (g's and h's) as the dependent variables: The δ's (β's and γ's) determine the asymptotics in (8) and they enter Z in an n- and p-dependent way (cf (19)). Obviously, the numerical specification of the selfconsistent g's and h's would be more straightforward; it fixes in practice only the 'non-dominant' components $r^{-2(p+1)/t}, r^{-2p/t}, \ldots, r^{2(q-2)/t}, r^{2(q-1)/t}$ of the potential $W(r)$ and completes the construction of an exact and elementary solution (8), (9), (20) and (7) to the Schrödinger equation (6).

In principle, assuming that the potential $W(r)$ is not varied (and the δ's may be fixed as well), a whole multiplet of some $M>1$ terminating solutions (8) may be constructed and characterised by the four remaining variable parameters—degree N, exponent s, energy $\eta(\sim g_{t-1} \sim \rho_{p+t}$ for $t \leqslant q$, or $\sim \delta_{p+t-q}$ for $q<t \leqslant 2q)$ and angular momentum l ($h_1 \sim l(l+1)+\text{constant}$). Nevertheless, for $t \neq q$ and $t \neq 2q$, the choice of N is unique and related to the potential by (13). Similarly, equation (12) fixes s when $p \neq 0$. In the 'exceptional' $t=q$ or $t=2q$ cases, the energy becomes a function of N or vice versa (cf the harmonic ($t=q$) and coulombic ($t=2q$) spectra which are given by (13) for $Q=q=1$). Similarly, for $p=0$, equation (12) prescribes the value

$$s=-\tfrac{3}{4}+\tfrac{1}{2}t\,[\text{constant}+(l+\tfrac{1}{2})^2]^{1/2}$$

as a function of l. Thus, when we consider the lth partial wave, the only variable parameter is in fact just the energy.

Let us assume for definiteness that $q \neq t \neq 2q$. Then, we may pick up the $M>1$ energies as roots of the selfconsistency equation (21*a*), $n=1$ (i.e., $M \leqslant N$), while the

$(2Q-2)$ free parameters must satisfy in general the $(Q-2)M$ remaining energy-dependent selfconsistencies (21*a*), $n \neq 1$. Thus, the multiplicity M cannot exceed the value $M_Q = \max(2Q-2)/(Q-2)$,

$$M_2 = N, \qquad M_3 = 4, \qquad M_4 = 3, \qquad M_Q = 2, \qquad Q > 4 \tag{22}$$

unless some of the selfconsistency equations appear to be independent of energy. After a similar discussion concerning $t = q$ or $t = 2q$, we may conclude that in practice, the multiplicity M is strongly restricted but, at the same time, the numerical selfconsistent construction of a terminating doublet (pair of the exact eigenstates) should be feasible for any force (4).

As a simple and interesting example, we may consider now the $p = q = 1$ harmonic oscillator with a repulsive core

$$-\frac{d^2}{dr^2}\psi(r) + \frac{l(l+1)}{r^2}\psi(r) + \left(\beta_1^2 r^2 + \frac{h_1'}{r^2} + \frac{h_2}{r^4} + \frac{\gamma_1^2}{r^6}\right)\psi(r) = E\psi(r). \tag{1*}$$

The form of this equation is symmetric with respect to the coordinate inversion $r \to 1/r$, and the elementary bound states are given of course by (8) and (20), with non-negative integer (angular momentum) l in $h_1 = h_1' + l(l+1)$, with the exponent $s = h_2/4\gamma_1 - \frac{1}{4}$ given by (12), and with the energies defined by the formula (13), $E = \beta_1(4N + h_2/\gamma_1)$.

The varying choices of N might correspond to the different states for the same potential. Indeed, the selfconsistency condition (14) becomes an N-dependent algebraic equation

$$\det\begin{pmatrix} C_1^{(0)} & B_1 & & \\ C_2^{(1)} & C_2^{(0)} & B_2 & \\ & \cdots & & \\ & & C_N^{(1)} & C_N^{(0)} \end{pmatrix} = 0 \tag{14*}$$

$$B_n = 4n\gamma_1, \qquad C_n^{(0)} = 4n^2 + 2n\left(\frac{h_2}{\gamma_1} - 2\right) - h_1 - 2\beta_1\gamma_1,$$

$$C_n^{(1)} = C_n^{(1)}(N) = 4\beta_1(N + 1 - n), \qquad n = 1, 2, \ldots N.$$

It depends on three variables only (say, $x = \beta_1\gamma_1$, $y = h_2/\gamma_1$ and $z = h_1 + 2\beta_1\gamma_1$), since we may fix one of the couplings by the scaling $r \to \text{constant} \times r$. Their proper choice may therefore provide the three elementary bound states with $N = N^{(1)}$, $N^{(2)}$ and $N^{(3)}$ at most.

The choice of the levels $N^{(i)}$ is not arbitrary. For illustration, let us pick up first the ground state, $N = N^{(1)} = 1$. Then, equation (14*) implies that $z = 2y$. As a further consequence, we obtain also that a_2 is identically zero for any N, so that the next possible choice of N is $N^{(2)} = 3$. In this case, we get simply the second restriction $y = -6$ from (14*). In accordance with (20), $N^{(2)} = 3$ defines the first excited state (with one nodal zero) so that the comparison with the unperturbed harmonic oscillators is possible in principle.

Since $a_4 = 0$ and $a_5 = 1536x^2(N-1)(N-3) \neq 0$ for $N > 3$, the second excited state must be represented by an infinite Taylor series $\varphi(r)$ $(N = \infty)$ which will not be investigated here. The third lowest choice of $N^{(3)} = 6$ fixes the value of $x = 24$ and corresponds to the third excited state. We see that the spectrum of the harmonic oscillator with a core ceases to be equidistant. In this way, our construction of the three elementary and exact bound states fixes the coupling constants in (1*), $\beta_1 = 1$ (scaling $r \to r/\sqrt{\beta_1}$), $\gamma_1 = 24$, $h_2 = -144$ and $h_1' = -60 - l(l+1)$. Similar constructions may be repeated for the other choices of $N^{(1)} > 1$ etc.

2.6. *The simplest potentials* ($Q < N$)

To illustrate (22), let us consider the first $Q = p + q = 2$ group of the non-trivial potentials (4) as listed in table 1. Their terminating eigenstates and explicit energies may be obtained by the same method as described by Singh *et al* (1978, 1979) for $p = 0$. Indeed, the conditions (12) and (13) are accompanied by the only 'Hill-determinant' condition (21*a*), $n = 1$, which is solvable by diagonalisation. For $p = 0$, $q = 2$ and $t = 1$ or 3, it gives a terminating multiplet for $M = M_2 = N$ energies. In all the other cases ($t = 2$, 4 or $p = q = 1$, $t = 1$, 2), the admissible energies are restricted by (13) as functions of an integer N, so that, *a priori*, we cannot construct more than $M = 4$ terminating states by the proper choice of all the four available couplings.

For the more complicated potentials ($Q > 2$) and $N > Q$, the pair of equations (20*a*) and (20*b*) ($a_{n+1}/a_1 = b_n(\rho_1, \ldots, \rho_n)$ and $a_{N-n}/a_N = c_{N-n}(\rho_{Q-1}, \ldots, \rho_{Q-n})$, respectively) is an overcomplete system equivalent to (14) when $n = 1, 2, \ldots, N-1$. Hence, we may interpret, say, the $n \geq Q$ part of (20) as mere definitions of $a_{Q+1}, \ldots, a_N$, and the identities $a_{n+1}/a_N = (a_{n+1}/a_1) \times (a_1/a_N)$ written in the form

$$c_{n+1}(\rho_{Q-1}, \ldots, \rho_{\max(1,Q+n-N+1)}) = b_n(\rho_1, \ldots, \rho_n) c_1(\rho_{Q-1}, \ldots, \rho_1), \qquad n = 1, 2, \ldots, Q-1 \tag{23}$$

as a coupled set of equations determining $\rho_1, \ldots \rho_{Q-1}$. These determinantal equations are to be solved numerically.

2.7. *The low-lying states* ($N \leq Q$)

We may expand (21*a*) and (21*b*) with respect to the last row of Z and Z^*, which gives

$$\sum_{j=i}^{\min(Q-1,N+i-1)} a_{N+i-j} C_{N+i}^{(j)} = 0, \qquad i = 0, 1, \cdots Q-2 \tag{24a}$$

and

$$\sum_{j=\max(1,i-N+1)}^{i} a_{i+1-j} C_i^{(j-1)} = -B_i a_{i+1}, \qquad i = 1, 2, \ldots, N-1,$$
$$= 0, \qquad i = N, N+1, \ldots, Q-1, \tag{24b}$$

Table 1. The class of potentials of the Singh type.

Q	p	q	t	$W(r)$	Physical restriction
2	0	2	1	$ar^{-2} + br^2 + cr^4 + dr^6$	$d > 0, a > -\frac{1}{4}$
2	0	2	2	$ar^{-2} + br^{-1} + cr + dr^2$	$d > 0, a > -\frac{1}{4}$
2	0	2	3	$ar^{-2} + br^{-4/3} + cr^{-2/3} + dr^{2/3}$	$d > 0, a > -\frac{1}{4}$
2	0	2	4	$ar^{-2} + br^{-3/2} + cr^{-1} + dr^{-1/2}$	$E < 0, a > -\frac{1}{4}$
2	1	1	1	$ar^{-6} + br^{-4} + cr^{-2} + dr^2$	$d > 0, a > 0$
2	1	1	2	$ar^{-4} + br^{-3} + cr^{-2} + dr^{-1}$	$E < 0, a > 0$

respectively. With respect to the linear ρ-dependence (19) of the C's, we may rewrite (24) as explicit formulae of the type

$$\rho_k = \tilde{b}_k(a_1, \ldots, a_{\min(k+1,N)}), \qquad k = 1, 2, \ldots, Q-1 \tag{25a}$$

$$\rho_{Q-m} = \tilde{c}_{Q-m}(a_N, \ldots, a_{\max(1,N-m)}), \qquad m = 1, 2, \ldots, Q-1. \tag{25b}$$

We may interpret (25) also as an inverse of the first $Q-1$ rows of (20) and (21) since $a_{n+1}/a_1 = b_n$ in formulae (a) or $a_{N-m}/a_N = c_{N-m}$ in (b) are linear functions of ρ_n or ρ_{Q-m}, respectively. Thus, the pair (25a), (25b) becomes equivalent to (14) for $N \leq Q$ and the trivial elimination of any $Q-N$ ρ's gives finally the coupled set

$$\tilde{b}_k(a_1, \ldots, a_{\min(N,k+1)}) = \tilde{c}_k(a_N, \ldots, a_{\max(1,N-Q+k)}) \tag{26}$$

$$k = k_1, k_2, \ldots, k_{N-1} \leq Q-1$$

of the $N-1$ nonlinear algebraic selfconsistency equations for the a's. In the rest of the paper, we shall consider only (26) in more detail.

2.8. *Polynomially anharmonic oscillators* ($p = 0$)

The structure of (26) is mainly influenced by the changing position of the anomalous n^2-dependent diagonal $C_n^{(p-1)}$ in Z for different p's. Without any significant loss of generality, we may consider the regular potentials only, with $p = 0$, $\rho \equiv \alpha$, $q = Q$, $\delta \equiv \beta$ and $C_n^{(p-1)} \equiv B_n$.

Equations (24a) and (13) may be written in the matrix form

$$\begin{pmatrix} a_N & a_{N-1} & \cdots & a_{N-Q+1} \\ 0 & a_N & \cdots & a_{N-Q+2} \\ \vdots & & & \\ 0 & \cdots & & a_N \end{pmatrix} \begin{pmatrix} C_N^{(0)} \\ C_{N+1}^{(1)} \\ \vdots \\ C_{N+Q-1}^{(Q-1)} \end{pmatrix}$$

$$= 4 \begin{pmatrix} 0 & a_{N-1} & 2a_{N-2} & \cdots & (Q-1)a_{N-Q+1} \\ 0 & 0 & a_{N-1} & \cdots & (Q-2)a_{N-Q+2} \\ \vdots & & & & \\ 0 & \cdots & & & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_Q \end{pmatrix}. \tag{27a}$$

An explicit inversion of the left-hand-side matrix gives immediately (25a),

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \ldots \end{pmatrix} = 4M^{(a)} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \ldots \end{pmatrix} \tag{28a}$$

where

$$M_{kk}^{(a)} = -(N+s), \qquad M_{kk+l}^{(a)} = [(-1)^{l+1}/a_N^l] \det F_l, \qquad M_{kk-l}^{(a)} = 0 \tag{29a}$$

$$F_l = \begin{pmatrix} a_{N-1} & 2a_{N-2} & \cdots & la_{N-l} \\ & & & \\ a_N & a_{N-1} & \cdots & a_{N-l+1} \\ 0 & a_N & \cdots & a_{N-l+2} \\ \vdots & & & \\ 0 & \cdots & a_N & a_{N-1} \end{pmatrix}, \qquad k, l = 1, 2 \ldots.$$

Similarly, we get

$$\begin{pmatrix} C_1^{(0)} \\ C_2^{(1)} \\ \vdots \\ C_Q^{(Q-1)} \end{pmatrix} = \begin{pmatrix} \alpha_1+4\beta_1(s+1) \\ \alpha_2+4\beta_2(s+1) \\ \vdots \\ \alpha_Q+4\beta_Q(s+1) \end{pmatrix} = -4M^{(b)} \begin{pmatrix} \frac{1}{4} \\ \beta_1 \\ \vdots \\ \beta_{Q-1} \end{pmatrix} \tag{28b}$$

$$M^{(b)} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \cdots \\ \vdots & & & \\ a_Q & \cdots & & a_1 \end{pmatrix}^{-1} \begin{pmatrix} B_1 a_2 & 0 & \cdots & 0 \\ B_2 a_3 & a_2 & \cdots & \\ \vdots & & & \\ B_Q a_{Q+1} & (Q-1)a_Q & & a_2 \end{pmatrix}$$

directly from (24b). Here, the n^2-anomaly enters in an explicit way and the matrix $M^{(b)}$ may easily be shown to read

$$M^{(b)}_{k+lk} = [(-1)^l/a_1^{l+1}] \det S_{l+1}, \qquad l=0,1,\ldots \tag{29b}$$

$$S_{l+1} = \begin{pmatrix} a_2 & a_1 & 0 & \cdots & 0 \\ 2a_3 & a_2 & a_1 & 0\cdots & 0 \\ 3a_4 & a_3 & a_2 & a_1 & \cdots \\ \vdots & & & & \\ (l+1)a_{l+2} & a_{l+1} & a_l & \cdots & a_2 \end{pmatrix}, \qquad k \neq 1$$

$$S_{l+1} = \begin{pmatrix} B_1 a_2 & a_1 & 0 & \cdots & 0 \\ B_2 a_3 & a_2 & a_1 & 0 & \cdots \\ \vdots & & \cdots & & \\ B_{l+1} a_{l+2} & a_{l+1} & & \cdots & a_2 \end{pmatrix}, \qquad k=1.$$

The analogous determinantal formulae may systematically be derived for any $p \neq 0$ but this will not be done here.

2.9. The $N=2$ example

For $p=0$ and $N=2$, equations (28a) and (28b) acquire simple forms

$$\alpha_i = -4\beta_i(2+s) + 4\sum_{m=1}^{Q-i} (-1)^{m+1}\left(\frac{a_1}{a_2}\right)^m \beta_{m+i} \qquad i=1,2,\ldots Q \tag{30a}$$

and

$$\alpha_i = -4\beta_i(1+s) + 4\sum_{m=1}^{i-1} (-1)^m \left(\frac{a_2}{a_1}\right)^m \beta_{i-m} + B_1\left(-\frac{a_2}{a_1}\right)^i \qquad i=1,2,\ldots Q \tag{30b}$$

respectively. Obviously, a comparison (26) of (30a) with (30b) at any index i gives the same polynomial equation

$$\sum_{j=1}^{Q} (-x)^j \beta_j = 2s + \tfrac{5}{2} \tag{31}$$

for the only unknown parameter $x = a_1/a_2$. The roots of (31) may be inserted into (30a) to determine all the couplings and energy $\eta(\sim\alpha_t,\ t \leq Q$, or $\sim\beta_{t-q},\ Q<t\leq 2Q)$.

Equation (31) reproduces the well known Laguerre polynomial harmonic case with $Q=1$ and $a_1=-(4s+5)/2\beta_1$. For $Q>1$, it encompasses and simplifies all the other known $N>1$ terminating solutions of Singh *et al* (with $N=Q=q=2$ and for $t=1$ in the 1978 paper and for $t=2$ in the 1979 Letter) and Magyari (1981, cf of his secular equation (20) for $Q=3$, $N=2$ and $t=1$).

The two different roots x of (31) correspond to the two different potentials $W(r)$ in (6) since both the constants α_{Q-1} and β_1 are linear functions of x. Hence, there are no $N=2$ multiplets—this was observed by Flessas and Das (1980) with $Q=2$.

2.10. *The $N=3$ example*

It is easy to infer from (24) that the restriction $M_Q \leqslant N-1$ (cf the preceding paragraph) must be added to (22) for any $N \geqslant 2$ and $Q>2$. Thus, the simplest example which permits the energy doublets necessitates N equal to three. With $p=0$, we have in (28)

$$\begin{aligned}\det F_{l+1} &= \det S_{l+1}^{(k\neq 1)} = a_2 R_l - 2a_1 a_3 R_{l-1}\\ \det S_{l+1}^{(k=1)} &= B_1 a_2 R_l - B_2 a_1 a_3 R_{l-1}\end{aligned} \tag{32}$$

where

$$R_l = \det\begin{pmatrix} a_2 & a_1 & 0 & \cdots & & 0\\ a_3 & a_2 & a_1 & 0 & \cdots & 0\\ 0 & a_3 & a_2 & a_1 & 0 & \cdots\\ \vdots & & & & & \\ 0 & \cdots & 0 & 0 & a_3 & a_2\end{pmatrix} = \det\begin{pmatrix} y & 0 & \cdots & 0\\ 0 & y^2 & 0 & \cdots\\ \vdots & & \cdots & \\ 0 & \cdots & & y^l\end{pmatrix}$$

$$\times\det\begin{pmatrix} z^0 & & \\ & z^{-1} & \\ & & \cdots\end{pmatrix}\times\det\begin{pmatrix} 2w & 1 & \cdots\\ 1 & 2w & \\ \cdots & & \end{pmatrix}$$

$$\times\det\begin{pmatrix} z & & \\ & z^2 & \\ & & \cdots\end{pmatrix}\times\det\begin{pmatrix} y^0 & & \\ & y^{-1} & \\ & & \cdots\end{pmatrix}$$

$$= y^l z^l U_l(w), \qquad y = a_3^{1/2}, \qquad z = a_1^{1/2}, \qquad 2w = a_2/yz$$

and $U_l(w)$ denotes the Chebyshev polynomial of the second kind (Gradshteyn and Ryzhik 1971). Hence, we get from (28a)

$$\alpha_i = -4\beta_i(3+s) + 8\sum_{l=1}^{Q-i}(-1)^{l+1}\left(\frac{a_1}{a_3}\right)^{l/2}\beta_{i+l}T_l(w) \qquad i=1,2,\ldots Q \tag{33a}$$

with the Chebyshev polynomials $T_l(w)$ of the first kind.

Similarly, from (28b) we arrive at the formula

$$\begin{aligned}\alpha_i = -4\beta_i(1+s) &+ 8\sum_{m=1}^{i-1}\left(\frac{a_3}{a_1}\right)^{m/2}(-1)^m T_m(w)\beta_{i-m}\\ &+(-1)^i\left(\frac{a_3}{a_1}\right)^{i/2}(2wB_1U_{i-1}(w) - B_2U_{i-2}(w)) \qquad i=1,2,\ldots Q.\end{aligned} \tag{33b}$$

Finally, the selfconsistency (26) for the only two unknown variables $w = a_2/(2a_1a_3)^{1/2}$

and $v=(a_1/a_3)^{1/2}$ may be written in the form

$$\sum_{m=1}^{Q}(-v)^m T_{|m-i|}(w)\beta_m=(2s+\tfrac{7}{2})T_i(w)-wU_{i-1}(w) \qquad i=i_1, i_2 \tag{34}$$

with any pair of indices $1\leqslant i_1<i_2\leqslant Q$.

When $Q=3$, we may eliminate w algebraically and specify v numerically as a root of an 18th degree polynomial.

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